The Correlated Binomial Distribution - Part I

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In August of 2004 Moody's published "The Moody's Correlated Binomial Default Distribution". Whereas the mathematics can be applied to any binomial event distribution where events are correlated Moody's emphasis was on modeling correlated defaults. In structured finance correlation is critical. Because the Moody's publication was sparce on the mathematical derivation of the correlated binomial this paper goes through the derivation step by step. As is usually the case we will start with a hypothetical problem, derive and discuss the mathematics needed to solve that problem and then apply the mathematics so actually solving the problem. The problem that we will be solving is the problem presented in Part I but with a correlation assumption. Let's begin...

Our Hypothetical Problem

In the PDF on The Binomial Distribution we considered the following problem...

Imagine that we have a bond portfolio that contains three bonds each with a principal balance of $1,000,000 and each with a maturity date one year hence. If at maturity the bond does not default we will receive the principal balance plus 15.00% simple interest. If at maturity the bond does default we will receive the recovery on that bond which is expected to be 40% of the principal balance. Each bond has an annual default probability of 0.10 and **defaults are independent**.

Question 1: What is the probability that our bond portfolio will experience zero defaults, one default, two defaults and three defaults?

Question 2: How much principal and interest can we expect to collect in one years time and what is our expected total return?

Given independent defaults the answer to the problem was...

Default	No Default	Combin	Indiv Prob	Total Prob	Payoff	Expected
0	3	1	0.7290	0.7290	3,450,000	2,515,050
1	2	3	0.0810	0.2430	2,700,000	656,100
2	1	3	0.0090	0.0270	1,950,000	52,650
3	0	1	0.0010	0.0010	1,200,000	1,200
Total		8		1.0000		3,225,000

Total expected return $= \frac{3{,}225{,}000}{3{,}000{,}000} - 1 = 7.50\%$

In Part II we will consider the case where defaults are correlated, which for credit risk problems is much more realistic than assuming independence. Our Part II task is to answer the hypothetical problem above assuming that default correlation is 0.30 rather then zero (i.e. independent defaults).

Setting Up The Problem

We will use the variable B_1 to represent the first bond in our portfolio, B_2 to represent the second bond in our portfolio and B_3 to represent the third bond in our portfolio. Note that order is not important. The variable X_1 represents the default indicator applicable to bond B_1 and will take the value of one if bond B_1 defaults and zero if bond B_1 does not default. The variable X_2 represents the default indicator applicable to bond B_2 and will take the value of one if bond B_2 defaults and zero if bond B_2 does not default. The variable X_3 represents the default indicator applicable to bond B_3 and will take the value of one if bond B_3 defaults and zero if bond B_3 does not default.

Per the hypothetical problem above the unconditional probability of default is...

$$p_0 = 0.10 \tag{1}$$

There Have Been No Prior Bond Defaults

The equation for the conditional default correlation between X_1 and X_2 given no prior defaults is...

$$\theta_{1,2} = \frac{\mathbb{E}[X_1\, X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2]}{\sqrt{Var(X_1)\, Var(X_2)}} \tag{2}$$

The expected value of X_1 is...

$$\begin{aligned}\mathbb{E}[X_1] &= (0 \times P[X_1 = 0]) + (1 \times P[X_1 = 1])\\ &= P[X_1 = 1]\\ &= p_0\end{aligned} \tag{3}$$

Using the mathematics of Equation (3) it can be shown that the expected values of X_2 and X_3 are...

$$\mathbb{E}[X_2] = p_0 \tag{4}$$

$$\mathbb{E}[X_3] = p_0 \tag{5}$$

The expected value of the square of X_1 is...

$$\begin{aligned}\mathbb{E}[X_1^2] &= (0^2 \times P[X_1 = 0]) + (1^2 \times P[X_1 = 1])\\ &= P[X_1 = 1]\\ &= p_0\end{aligned} \tag{6}$$

Using the mathematics of Equation (6) it can be shown that the expected values of the squares of X_2 and X_3 are...

$$\mathbb{E}[X_2] = p_0 \tag{7}$$

$$\mathbb{E}[X_3] = p_0 \tag{8}$$

The variance is defined as the second moment of the distribution of X minus the square of the first moment of the distribution of X. Using Equations (3) and (6) above the variance of X_1 is...

$$\begin{aligned}Var(X_1) &= \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2\\ &= p_0 - p_0^2\\ &= p_0\,(1 - p_0)\end{aligned} \tag{9}$$

Using the mathematics of Equation (9) it can be shown that the variances of X_2 and X_3 are...

$$Var(X_2) = p_0\,(1 - p_0) \tag{10}$$

$$Var(X_3) = p_0\,(1 - p_0) \tag{11}$$

The expected value of the product of X_1 and X_2 is...

$$\begin{aligned}\mathbb{E}[X_1\, X_2] &= (0 \times 0 \times P[X_1 = 0 \cap X_2 = 0]) + (1 \times 0 \times P[X_1 = 1 \cap X_2 = 0]) +\\ &\quad (0 \times 1 \times P[X_1 = 0 \cap X_2 = 1]) + (1 \times 1 \times P[X_1 = 1 \cap X_2 = 1])\\ &= P[X_1 = 1 \cap X_2 = 1]\\ &= P[X_1 = 1]\, P[X_2 = 1 | X_1 = 1]\end{aligned} \tag{12}$$

Note that per Equation (12) above the expected value of the product of X_1 and X_2 is the probability that bonds B_1 and B_2 both default. The equation for this compound probability is the probability that bond B_1 defaults times the probability that bond B_2 defaults given that bond B_1 defaults.

We can equate equations Equations (2) and (12) and solve for the conditional probability that bond B_2 will default given that bond B_1 has already defaulted. The equation for the conditional probability that $X_2 = 1$ given that $X_1 = 1$ is...

$$\begin{aligned}
\mathbb{E}[X_1\, X_2] &= \theta_{1,2} \sqrt{Var(X_1)\, Var(X_2)} + \mathbb{E}[X_1]\, \mathbb{E}[X_2] \\
P[X_1 = 1]\, P[X_2 = 1|X_1 = 1] &= \theta_{1,2} \sqrt{p_0(1-p_0)\, p_0(1-p_0)} + p_0^2 \\
p_0\, P[X_2 = 1|X_1 = 1] &= \theta_{1,2}\, p\, (1-p_0) + p_0^2 \\
P[X_2 = 1|X_1 = 1] &= \theta_{1,2}\, (1-p-0) + p_0 \qquad (13)
\end{aligned}$$

For the sake of convenience we will define the variable p_1 to be the conditional probability that bond B_2 defaults given that bond B_1 has already defaulted. Using this definition we can rewrite Equation (13) above as...

$$\begin{aligned}
p_1 &= P[X_2 = 1|X_1 = 1] \\
&= \theta_{1,2}\, (1-p_0) + p_0 \\
&= p_0\, (1-\theta_{1,2}) + \theta_{1,2} \qquad (14)
\end{aligned}$$

We can now combine Equations (12) and (14) and show that the expected value of the product of X_1 and X_2 can be rewritten is...

$$\begin{aligned}
\mathbb{E}[X_1\, X_2] &= P[X_1 = 1]\, P[X_2 = 1|X_1 = 1] \\
&= p_0\, p_1 \qquad (15)
\end{aligned}$$

Using the mathematics of Equation (15) it can be shown that the expected values of the product of X_1 and X_3 and the product of X_2 and X_3 are...

$$\mathbb{E}[X_1\, X_3] = p_0\, p_1 \qquad (16)$$
$$\mathbb{E}[X_2\, X_3] = p_0\, p_1 \qquad (17)$$

There Has Been One Prior Bond Default

The equation for the conditional default correlation between X_2 and X_3 given that bond B_1 has already defaulted (i.e. $X_1 = 1$) is...

$$\theta_{2,3} = \frac{\mathbb{E}[X_2\, X_3|X_1 = 1] - \mathbb{E}[X_2|X_1 = 1]\, \mathbb{E}[X_3|X_1 = 1]}{\sqrt{Var(X_2|X_1 = 1)\, Var(X_3|X_1 = 1)}} \qquad (18)$$

The expected value of X_2 given that bond B_1 has already defaulted (uses Equation (14)) is...

$$\begin{aligned}
\mathbb{E}[X_2|X_1 = 1] &= (0 \times P[X_2 = 0|X_1 = 1]) + (1 \times P[X_2 = 1|X_1 = 1]) \\
&= P[X_2 = 1|X_1 = 1] \\
&= p_1 \qquad (19)
\end{aligned}$$

The expected value of the square of X_2 given that bond B_1 has already defaulted (uses Equation (14)) is...

$$\begin{aligned}
\mathbb{E}[X_2^2|X_1 = 1] &= (0^2 \times P[X_2 = 0|X_1 = 1]) + (1^2 \times P[X_2 = 1|X_1 = 1]) \\
&= P[X_2 = 1|X_1 = 1] \\
&= p_1 \qquad (20)
\end{aligned}$$

Using Equations (19) and (20) the variance of X_2 given that bond B_1 has already defaulted is...

$$\begin{aligned}
Var(X_2|X_1 = 1) &= \mathbb{E}[X_2^2|X_1 = 1] - (\mathbb{E}[X_2|X_1 = 1])^2 \\
&= p_1 - p_1^2 \\
&= p_1\, (1-p_1) \qquad (21)
\end{aligned}$$

The expected value of the product of X_2 and X_3 given that bond B_1 has already defaulted is...

$$\begin{aligned}
\mathbb{E}[X_2\,X_3|X_1=1] &= (0\times 0\times P[X_2=0\cap X_3=0|X_1=1])+(1\times 0\times P[X_2=1\cap X_3=0|X_1=1])+\\
&\quad (0\times 1\times P[X_2=0\cap X_3=1|X_1=1])+(1\times 1\times P[X_2=1\cap X_3=1|X_1=1])\\
&= P[X_2=1\cap X_3=1|X_1=1]\\
&= P[X_2=1|X_1=1]\,P[X_3=1|X_1=1,X_2=1] \qquad (22)
\end{aligned}$$

Note that per Equation (22) above the expected value of the product of X_2 and X_3 given that $X_1=1$ is the probability that bonds B_2 and B_3 both default given that bond B_1 has already defaulted. The equation for this compound probability is the probability that bond B_2 will default given that bond B_1 has already defaulted times the probability that bond B_3 will default given that both bonds B_1 and B_2 have already defaulted.

We can now equate equations Equations (18) and (22) and solve for the conditional probability that bond B_3 will default given that bonds B_1 and B_2 have already defaulted. The equation for the conditional probability that $X_3=1$ given that $X_1=1$ and $X_2=1$ is...

$$\begin{aligned}
\mathbb{E}[X_3\,X_2|X_1=1] &= \theta_{2,3}\sqrt{Var(X_2|X_1=1)\,Var(X_3|X_1=1)}+\mathbb{E}[X_2|X_1=1]\,\mathbb{E}[X_3|X_1=1]\\
P[X_2=1|X_1=1]\,P[X_3=1|X_1=1,X_2=1] &= \theta_{2,3}\sqrt{p_1(1-p_1)\,p_1(1-p_1)}+p_1^2\\
p_1\,P[X_3=1|X_1=1,X_2=1] &= \theta_{2,3}\,p_1(1-p_1)+p_1^2\\
P[X_3=1|X_1=1,X_2=1] &= \theta_{2,3}\,(1-p_1)+p_1 \qquad (23)
\end{aligned}$$

For the sake of convenience we will define the variable p_2 to be the conditional probability that bond B_3 defaults given that bonds B_1 and B_2 have already defaulted. Using this definition we can rewrite Equation (23) above as...

$$\begin{aligned}
p_2 &= P[X_3=1|X_1=1,X_2=1]\\
&= \theta_{2,3}\,(1-p_1)+p_1\\
&= p_1\,(1-\theta_{2,3})+\theta_{2,3} \qquad (24)
\end{aligned}$$

Using Equations (14) and (24) above the expected value of the product of X_1, X_2 and X_3 is...

$$\begin{aligned}
\mathbb{E}[X_1X_2X_3] &= (0\times 0\times 0\times P[X_1=0\cap X_2=0\cap X_3=0])+(1\times 0\times 0\times P[X_1=1\cap X_2=0\cap X_3=0])\\
&\quad (1\times 1\times 0\times P[X_1=1\cap X_2=1\cap X_3=0])+(1\times 0\times 1\times P[X_1=1\cap X_2=0\cap X_3=1])\\
&\quad (0\times 1\times 0\times P[X_1=0\cap X_2=1\cap X_3=0])+(0\times 1\times 1\times P[X_1=0\cap X_2=1\cap X_3=1])\\
&\quad (0\times 0\times 1\times P[X_1=0\cap X_2=0\cap X_3=1])+(1\times 1\times 1\times P[X_1=1\cap X_2=1\cap X_3=1])\\
&= P[X_1=1\cap X_2=1\cap X_3=1]\\
&= P[X_1=1]\,P[X_2=1|X_1=1]\,P[X_3=1|X_1=1,X_2=1]\\
&= p_0\,p_1\,p_2 \qquad (25)
\end{aligned}$$

The Answer To Our Hypothetical Problem

Note that per Equations (3), (4) and (5) above the probability that $X_i=1$ (i.e. Asset A_i defaults) is $\mathbb{E}[X_i]$ and accordingly the probability that $X_i=0$ (i.e. Asset A_i does not default) is $1-\mathbb{E}[X_i]$.

Question one to our problem is to find the probabilities that our bond portfolio will experience zero defaults, one default, two defaults and three defaults. Using Equations (3), (4), (5), (15), (16), (17) and (25) above the equation for the probability that our portfolio will experience zero defaults ($X_1=0$, $X_2=0$, $X_3=0$) is...

$$\begin{aligned}
P[0] &= \mathbb{E}[(1-X_1)(1-X_2)(1-X_3)]\\
&= \mathbb{E}[(1-X_1)(1-X_2-X_3+X_2X_3)]\\
&= \mathbb{E}[1-X_1-X_2-X_3+X_1X_2+X_1X_3+X_2X_3-X_1X_2X_3]\\
&= 1-\mathbb{E}[X_1]-\mathbb{E}[X_2]-\mathbb{E}[X_3]+\mathbb{E}[X_1X_2]+\mathbb{E}[X_1X_3]+\mathbb{E}[X_2X_3]-\mathbb{E}[X_1X_2X_3]\\
&= 1-p_0-p_0-p_0+p_0\,p_1+p_0\,p_1+p_0\,p_1-p_0\,p_1\,p_2\\
&= 1-3\,p_0+3\,p_0\,p_1-p_0\,p_1\,p_2 \qquad (26)
\end{aligned}$$

Using Equations (3), (15), (16) and (25) above the equation for the probability that our portfolio will experience one default ($\{X_1 = 1,\ X_2 = 0,\ X_3 = 0\}$ ...or... $\{X_1 = 0,\ X_2 = 1,\ X_3 = 0\}$ ...or... $\{X_1 = 0,\ X_2 = 0,\ X_3 = 1\}$) is...

$$\begin{aligned} P[1] &= \mathbb{E}[X_1(1 - X_2)(1 - X_3)] \\ &= \mathbb{E}[X_1(1 - X_2 - X_3 + X_2 X_3)] \\ &= \mathbb{E}[X_1 - X_1 X_2 - X_1 X_3 + X_1 X_2 X_3)] \\ &= \mathbb{E}[X_1] - \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1 X_3] + \mathbb{E}[X_1 X_2 X_3] \\ &= p_0 - p_0\, p_1 - p_0\, p_1 + p_0\, p_1\, p_2 \\ &= p_0 - 2\, p_0\, p_1 + p_0\, p_1\, p_2 \end{aligned} \tag{27}$$

Using Equations (15) and (25) above the equation for the probability that our portfolio will experience two defaults ($\{X_1 = 1,\ X_2 = 1,\ X_3 = 0\}$ ...or... $\{X_1 = 1,\ X_2 = 0,\ X_3 = 1\}$ ...or... $\{X_1 = 0,\ X_2 = 1,\ X_3 = 1\}$) is...

$$\begin{aligned} P[2] &= \mathbb{E}[X_1 X_2(1 - X_3)] \\ &= \mathbb{E}[X_1 X_2 - X_1 X_2 X_3] \\ &= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1 X_2 X_3] \\ &= p_0\, p_1 - p_0\, p_1\, p_2 \end{aligned} \tag{28}$$

Using Equation (25) above, the equation for the probability that our portfolio will experience three defaults ($X_1 = 1$, $X_2 = 1,\ X_3 = 1$) is...

$$\begin{aligned} P[3] &= \mathbb{E}[X_1 X_2 X_3] \\ &= p_0\, p_1\, p_2 \end{aligned} \tag{29}$$

Per Equations (1), (14) and (24) above, the values of p_0, p_1 and p_2 are...

$$p_0 = 0.1000 \tag{30}$$

$$p_1 = p_0\,(1 - \theta_{1,2}) + \theta_{1,2} = 0.10(1 - 0.30) + 0.30 = 0.3700 \tag{31}$$

$$p_2 = p_1\,(1 - \theta_{2,3}) + \theta_{2,3} = 0.37(1 - 0.30) + 0.30 = 0.5590 \tag{32}$$

Given correlated defaults the answer to the problem is...

Default	No Default	Combin	Indiv Prob	Total Prob	Payoff	Expected	Reference
0	3	1	0.7903	0.7903	3,450,000	2,726,594	Equation (26)
1	2	3	0.0467	0.1400	2,700,000	378,132	Equation (27)
2	1	3	0.0163	0.0490	1,950,000	95,454	Equation (28)
3	0	1	0.0207	0.0207	1,200,000	24,820	Equation (29)
Total		8		1.0000		3,225,000	

Total expected return $= \frac{3{,}225{,}000}{3{,}000{,}000} - 1 = 7.50\%$

Note that in both the independent case and the correlated case the expected cash flow and total return of \$3,225,000 and 7.50%, respectively, is the same in both cases. The difference is the shape of the probability distribution which in the correlated case puts much more weight on bad events (2 and 3 defaults) than does the independent case. The table below presents the respective probability distributions...

Defaults	Independent	Correlated
0	0.7290	0.7903
1	0.2430	0.1400
2	0.0270	0.0490
3	0.0010	0.0207
Total	1.0000	1.0000

To ignore correlation in credit risk management (and in other areas of valuation) is not only bad practice it is gross imcompetence.